Braided Quantum Field Theory

Richard Szabo

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

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Outline

- Introduction/Motivation
- Braided Gauge Symmetry
- Braided L_∞-Algebras & Braided Field Theories
- Braided BV Quantization

with M. Dimitrijević Ćirić, G. Giotopoulos, N. Konjik, H. Nguyen, V. Radovanović, A. Schenkel, M. Toman

[arXiv: 2103.08939 , 2107.02532 , 2112.00541 , 2204.06448] + ...

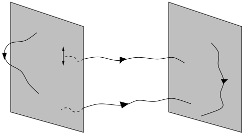
Noncommutative Field Theory

- **Noncommutative field theories** appear as effective theories in many physical scenarios, and in particular as frameworks for models of quantum gravity

- QFTs may teach us something about the interplay between quantum mechanics and short-distance spacetime structure

- **Example:** In constant NS–NS B-field backgrounds, open string interactions in CFT correlation functions captured by Moyal-Weyl star-product:

$$f \star g \; = \; \cdot \exp\big(\tfrac{\mathrm{i}}{2}\,\theta^{\mu\nu}\,\partial_\mu \otimes \partial_\nu\big)(f \otimes g) \quad , \quad \theta \; = \; B^{-1}$$



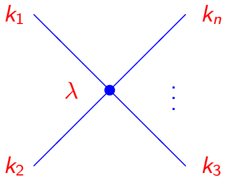
Low-energy dynamics described by noncommutative gauge theory

(Douglas & Hull '97; Ardalan, Arfaei & Sheikh-Jabbari '98; Chu & Ho '98; Schomerus '99; Seiberg & Witten '99; . . .)

UV/IR Mixing

- These theories are plagued by the problem of **UV/IR mixing**:

$$\widetilde{\phi}(k)\,\widetilde{\phi}(q) \longrightarrow \widetilde{\phi}(k)\,\widetilde{\phi}(q)\ \mathrm{e}^{\,\mathrm{i}\,k\times q}\ , \qquad k\times q = \tfrac{1}{2}\,k_\mu\,\theta^{\mu\nu}\,q_\nu$$



$$= \lambda\ \mathrm{e}^{\,\mathrm{i}\sum_{I<J} k_I\times k_J}$$

with $k_1+k_2+\ldots+k_n = 0$; effective at energies E with $E\sqrt{\theta} \ll 1$

- **Non-planar graphs:** UV cutoff Λ $\Longrightarrow$ Effective IR cutoff

$$\Lambda_0 = \frac{1}{\theta\,\Lambda}$$

(Minwalla, Van Raamsdonk & Seiberg '99)

- **The field theory cannot be renormalized!!!**

UV/IR Mixing

- **Grosse–Wulkenhaar model:** Real Euclidean scalar $\lambda\,\phi^{\star 4}_{2d}$-theory in background harmonic oscillator potential:

$$\Box \longmapsto \Box + \tfrac{1}{2}\,\omega^2\,\tilde{x}^2 \quad , \quad \tilde{x} = 2\,\theta^{-1}\cdot x$$

- QFT symmetric under Fourier transformation of fields: $k_\mu \leftrightarrow \widetilde{x}_\mu$
  Renormalizable to all orders in λ
  (Langmann & Sz '02; Grosse & Wulkenhaar '04; Rivasseau *et al.* '05; . . .)

- **In this talk:** A new approach to renormalizable noncomm QFT by modifying the *path integral* directly (not the classical theory)
  — this is called braided quantum field theory

- Renormalization properties of braided QFT very different
  — UV/IR mixing seems far less severe and maybe even absent
  (Oeckl '00; Balachandran *et al.* '06; Bu *et al.* '06; Fiore & Wess '07; . . .)

Braided Quantum Field Theory

- **Higher structures:** Deform L_∞-algebra description of (noncomm) field theories: **Braided L_∞-algebras** construct braided field theories equivariant under a triangular Hopf algebra action, with braided noncommutative fields (Dimitrijević Ćirić, Giotopoulos, Radovanović & Sz '21)

- Notion of braided gauge symmetry is not new — kinematical aspects of this idea have appeared before (Brzezinski & Majid '92; . . .)
  — ideas and techniques borrowed from twisted noncommutative gravity (Aschieri *et al.* '05; . . .)

- Oeckl's algebraic approach to braided QFT based on braided Wick's Theorem and Gaussian integration — (Oeckl '99; Sasai & Sasakura '07) but does not treat theories with gauge symmetries

- **Goal:** Apply modern incarnation of Batalin-Vilkovisky (BV) quantization (à la Costello-Gwilliam), in a braided version which completely captures perturbative braided QFT with explicit computations of correlation functions (Nguyen, Schenkel & Sz '21)

Drinfel'd Twist Deformation

- Let $\mathcal{F} \ = \ \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha \in U\Gamma(TM) \otimes U\Gamma(TM)$ be a Drinfel'd twist;
  e.g. Moyal-Weyl twist $\mathcal{F} = \exp\big(-\frac{\mathrm{i}}{2}\, \theta^{\mu\nu}\, \partial_\mu \otimes \partial_\nu \big)$
- If $\mathcal{A}$ is a $U\Gamma(TM)$-module algebra (functions, forms, tensors on M), then $\Gamma(TM)$ acts on $\mathcal{A}$ via Lie derivative and Leibniz rule
- Deform product on $\mathcal{A}$ into a star-product:

$$a \star b \ = \ \cdot\, \mathcal{F}^{-1}(a \otimes b) \ = \ \bar{\mathrm{f}}^\alpha(a) \cdot \bar{\mathrm{f}}_\alpha(b)$$

- Defines noncommutative algebra $\mathcal{A}_\star$ carrying representation of twisted Hopf algebra $U_{\mathcal{F}}\Gamma(TM)$
- If $\mathcal{A}$ is commutative, then $\mathcal{A}_\star$ is braided-commutative:

$$a \star b \ = \ R_\alpha(b) \star R^\alpha(a)$$

$\mathcal{R} \ = \ \mathcal{F}^{-2} \ = \ R^\alpha \otimes R_\alpha \ = \$ triangular $\mathcal{R}$-matrix

Braided Gauge Symmetry

- Braided Lie algebra $\Omega^0_\star(M,\mathfrak{g})$: $[\lambda_1,\lambda_2]^\star_{\mathfrak{g}} \ := \ [-,-]_{\mathfrak{g}} \circ \mathcal{F}^{-1}(\lambda_1 \otimes \lambda_2)$

- Braided antisymmetry and braided Jacobi identity:

$$[\lambda_1,\lambda_2]^\star_{\mathfrak{g}} \ = \ -[R_\alpha \lambda_2, R^\alpha \lambda_1]_\star$$

$$[\lambda_1,[\lambda_2,\lambda_3]^\star_{\mathfrak{g}}]^\star_{\mathfrak{g}} \ = \ [[\lambda_1,\lambda_2]^\star_{\mathfrak{g}},\lambda_3]^\star_{\mathfrak{g}} + [R_\alpha(\lambda_2),[R^\alpha(\lambda_1),\lambda_3]^\star_{\mathfrak{g}}]^\star_{\mathfrak{g}}$$

(Woronowicz '89; Majid '93; . . .)

- For matrix $\mathfrak{g}$: $[\lambda_1,\lambda_2]^\star_{\mathfrak{g}} \ = \ \lambda_1 \star \lambda_2 - R_\alpha(\lambda_2) \star R^\alpha(\lambda_1) \ \neq \ [\lambda_1 \overset{\star}{,} \lambda_2]_{\mathfrak{g}}$

- Braided gauge fields, matter fields $A \in \Omega^1_\star(M,\mathfrak{g})$, $\phi \in \Omega^p_\star(M,W)$ transform in (left) braided representations:

$$\delta^\star_\lambda \phi \ = \ -\lambda \star \phi \qquad , \qquad \delta^\star_\lambda A \ = \ \mathrm{d}\lambda - [\lambda, A]^\star_{\mathfrak{g}}$$

Braided Gauge Symmetry

- Braided gauge transformations satisfy braided Leibniz rule:

$$\delta^\star_\lambda(\phi \otimes A) \ = \ \delta^\star_\lambda \phi \otimes A + R_\alpha \phi \otimes \delta^\star_{R^\alpha \lambda} A$$

- Braided gauge transformations close a braided Lie algebra:

$$\big[\delta^\star_{\lambda_1}, \delta^\star_{\lambda_2}\big]^\star \ := \ \delta^\star_{\lambda_1}\, \delta^\star_{\lambda_2} - \delta^\star_{R_\alpha \lambda_2}\, \delta^\star_{R^\alpha \lambda_1} \ = \ \delta^\star_{[\lambda_1, \lambda_2]^\star_{\mathfrak{g}}}$$

- Braided left/right covariant derivatives:

$$\mathrm{d}^A_{\star \mathrm{L}} \phi \ := \ \mathrm{d}\phi + A \wedge_\star \phi \qquad , \qquad \mathrm{d}^A_{\star \mathrm{R}} \phi \ := \ \mathrm{d}\phi + R_\alpha(A) \wedge_\star R^\alpha(\phi)$$

Braided covariance: $\delta^\star_\lambda\big(\mathrm{d}^A_{\star \mathrm{L,R}} \phi\big) \ = \ -\lambda \star \big(\mathrm{d}^A_{\star \mathrm{L,R}} \phi\big)$

- Braided curvature:

$$F^\star_A \ := \ \mathrm{d}A + \tfrac{1}{2}\,[A, A]^\star_{\mathfrak{g}} \quad , \quad \delta^\star_\lambda F^\star_A \ = \ -[\lambda, F^\star_A]^\star_{\mathfrak{g}}$$

L_∞-Algebras of Classical Field Theories

- L_∞-algebras organise gauge symmetries and dynamics:

$$\begin{array}{ccccccccccc} \cdots \to & V_0 & \to & V_1 & \to & V_2 & \to & V_3 & \to & \cdots \\ \cdots & \text{gauge par.} & & \text{fields} & & \text{field eqs.} & & \text{Noether ids.} & & \cdots \end{array}$$

(Hohm & Zwiebach '17; Jurčo, Raspollini, Sämann & Wolf '18)

- Multilinear maps $\ell_n : \wedge^n V \longrightarrow V$ on $V = \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots$:

$$\ell_1(\ell_1(v)) = 0 \qquad (V, \ell_1) \text{ is a cochain complex}$$

$$\ell_1(\ell_2(v,w)) = \ell_2(\ell_1(v),w) \pm \ell_2(v,\ell_1(w)) \quad \ell_1 \text{ is a derivation of } \ell_2$$

$$\ell_2(v,\ell_2(w,u)) + \text{cyclic} = (\ell_1 \circ \ell_3 \pm \ell_3 \circ \ell_1)(v,w,u) \quad \text{Jacobi up to homotopy}$$

plus "higher homotopy Jacobi identities"

- L_∞-algebras are homotopy coherent generalizations of Lie algebras

- Graded inner product $\langle -,- \rangle : V \times V \longrightarrow \mathbb{R}$ gives cyclic structure:

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n)\rangle = \pm \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1})\rangle$$

Braided L_∞-Algebras of Braided Field Theories

- If $(V,\{\ell_n\})$ is a classical L_∞-algebra in the category of $U\Gamma(TM)$-modules, then $(V,\{\ell_n^\star\})$ is a braided L_∞-algebra in the category of $U_{\mathcal{F}}\Gamma(TM)$-modules, where

$$\ell_n^\star(v_1\wedge\cdots\wedge v_n) \ := \ \ell_n(v_1\wedge_\star\cdots\wedge_\star v_n)$$

- Braided graded antisymmetry:

$$\ell_n^\star(\dots,v,v',\dots) \ = \ -(-1)^{|v|\,|v'|}\ \ell_n^\star(\dots,R_\alpha(v'),R^\alpha(v),\dots)$$

$+$ braided homotopy Jacobi identities (unchanged for $n=1,2$)

- Braided L_∞-algebras are homotopy coherent generalizations of braided Lie algebras

- Cyclic inner product: $\langle-,-\rangle_\star \ := \ \langle-,-\rangle\circ\mathcal{F}^{-1}$

Braided L_∞-Algebras of Braided Field Theories

- Braided gauge transformations $\delta^\star_\lambda A \;=\; \ell^\star_1(\lambda)+\ell^\star_2(\lambda,A)+\cdots$ close a braided Lie algebra under braided commutator $[-,-]^\star$
- Braided field eqs $F^\star_A \;=\; \ell^\star_1(A)-\frac{1}{2}\,\ell^\star_2(A,A)+\cdots$ are covariant:

$$\delta^\star_\lambda F^\star_A \;=\; \ell^\star_2(\lambda,F^\star_A)+\frac{1}{2}\left(\ell^\star_3(\lambda,F^\star_A,A)-\ell^\star_3(\lambda,A,F^\star_A)\right)+\cdots$$

- Braided Noether ids from weighted sum over all braided homotopy identities on (A^n):

$$\begin{aligned}\mathcal{I}^\star_A F^\star_A \;&=\; \ell^\star_1(F^\star_A)+\frac{1}{2}\left(\ell^\star_2(F^\star_A,A)-\ell^\star_2(A,F^\star_A)\right)\\ &\quad+\frac{1}{3!}\,\ell^\star_1\left(\ell^\star_3(A^3)\right)+\frac{1}{4}\left(\ell^\star_2(\ell^\star_2(A^2),A)-\ell^\star_2(A,\ell^\star_2(A^2))\right)+\cdots\;\equiv\;0\end{aligned}$$

- Action: $S^\star \;=\; \frac{1}{2}\,\langle A,\ell^\star_1(A)\rangle_\star-\frac{1}{3!}\,\langle A,\ell^\star_2(A,A)\rangle_\star+\cdots$

$$\delta S^\star \;=\; \langle \delta A,F^\star_A\rangle_\star \quad , \quad \delta^\star_\lambda S^\star \;=\; -\langle\lambda,\mathcal{I}^\star_A F^\star_A\rangle_\star$$

- Systematic constructions of **new** noncomm. field theories with no new degrees of freedom, good classical limit, and some "surprises"

Braided Noncommutative Yang-Mills Theory

- Starting from $S = \frac{1}{2}\int \mathrm{Tr}_{\mathfrak{g}}(F_A \wedge * F_A)$ for $A \in \Omega^1(\mathbb{R}^d, \mathfrak{g})$, Yang-Mills L_∞-algebra is not a dg Lie algebra ($\ell_3(A^3) \neq 0$)

- Braided Yang-Mills L_∞-algebra gives braided field equations:

$$\tfrac{1}{2}\left(\mathrm{d}^A_{\star\mathrm{L}} * F^\star_A + \mathrm{d}^A_{\star\mathrm{R}} * F^\star_A\right) + \tfrac{1}{6}\left[R_\alpha(A), *[R^\alpha(A), A]^\star_{\mathfrak{g}}\right]^\star_{\mathfrak{g}} - \tfrac{1}{12}\left[A, *[A, A]^\star_{\mathfrak{g}}\right]^\star_{\mathfrak{g}} + \tfrac{(-1)^d}{12}\left[*[A, A]^\star_{\mathfrak{g}}, A\right]^\star_{\mathfrak{g}} = 0$$

  Covariant, classical limit gives Yang-Mills equations $\mathrm{d}^A * F_A = 0$

- Action:

$$S^\star = \frac{1}{2}\int \mathrm{Tr}_{\mathfrak{g}}\left(F^\star_A \wedge_\star * F^\star_A\right) + \frac{1}{24}\int \mathrm{Tr}_{\mathfrak{g}}\left([A, R_\alpha(A)]^\star_{\mathfrak{g}} \wedge_\star *[R^\alpha(A), A]^\star_{\mathfrak{g}}\right) - [A, A]^\star_{\mathfrak{g}} \wedge_\star *[A, A]^\star_{\mathfrak{g}}\,)$$

  Gauge invariant with good classical limit

- Noether ids: complicated modification of $\mathrm{d}^A(\mathrm{d}^A * F_A) = 0$ ...

  — New deformation of Yang-Mills theory

Braided BV Formalism

- $(V, \{\ell_n^\star\}, \langle -,- \rangle_\star)$ — braided cyclic L_∞-algebra
- Braided symmetric algebra $\mathrm{Sym}_{\mathcal{R}} V[2]$:
$$\varphi\,\psi \ = \ (-1)^{|\varphi|\,|\psi|}\,(R_\alpha\psi)\,(R^\alpha\varphi)$$
- Extended braided L_∞-algebra $\{\ell_n^{\star\mathrm{ext}}\}$ on $(\mathrm{Sym}_{\mathcal{R}} V[2])\otimes V$:
$$\ell_1^{\star\mathrm{ext}}(a\otimes v) \ = \ a\otimes \ell_1^\star(v)$$
$$\ell_2^{\star\mathrm{ext}}(a_1\otimes v_1, a_2\otimes v_2) \ = \ \pm\, a_1\,(R_\alpha a_2)\,\ell_2^\star(R^\alpha v_1, v_2) \quad \text{etc.}$$
- Choose dual bases $\varepsilon_\alpha \in V$, $\varrho^\alpha \in V^* \simeq V[3]$ and 'contracted coordinate functions' $\mathsf{a} \ = \ \varrho^\alpha\otimes\varepsilon_\alpha \ \in \ (\mathrm{Sym}_{\mathcal{R}} V[2])\otimes V$
- Braided BV Action $S^\star_{\mathrm{BV}} \in \mathrm{Sym}_{\mathcal{R}} V[2]$:
$$S^\star_{\mathrm{BV}} \ = \ S_0^\star + S^\star_{\mathrm{int}} \ = \ \tfrac{1}{2}\,\langle \mathsf{a}, \ell_1^{\star\mathrm{ext}}(\mathsf{a})\rangle_{\star\mathrm{ext}} + \tfrac{1}{3!}\,\langle \mathsf{a}, \ell_2^{\star\mathrm{ext}}(\mathsf{a},\mathsf{a})\rangle_{\star\mathrm{ext}} + \cdots$$

Braided BV Formalism

- (Classical) Master Equation: $\{S^\star_{\rm BV}, S^\star_{\rm BV}\}_\star \ = \ 0$,
  with bracket $\{\varphi,\psi\}_\star \ = \ \langle\varphi,\psi\rangle_\star \, \mathbb{1}$ for $\varphi,\psi\in V[2]$

- $Q^2 \ = \ 0$ where $Q \ = \ \ell_1^\star + \{S^\star_{\rm int}, -\}_\star$

- Classical observables $\big(\mathrm{Sym}_{\mathcal{R}} V[1]^* \simeq \mathrm{Sym}_{\mathcal{R}} V[2], Q, \{-,-\}_\star\big)$
  form a braided P_0-algebra:

$$-Q\{\varphi,\psi\}_\star \ = \ \{Q\varphi,\psi\}_\star + (-1)^{|\varphi|}\,\{\varphi,Q\psi\}_\star \qquad \text{Leibniz rule}$$

$$\{\varphi,\psi\}_\star \ = \ (-1)^{|\varphi|\,|\psi|}\,\{R_\alpha\psi, R^\alpha\varphi\}_\star \qquad \text{braided symmetric}$$

$$\{\varphi,\{\psi,\chi\}_\star\}_\star \ = \ \pm\{R_\alpha\psi,\{R_\beta\chi, R^\beta\,R^\alpha\varphi\}_\star\}_\star \pm \{R_\beta\,R_\alpha\chi,\{R^\beta\varphi, R^\alpha\psi\}_\star\}_\star \qquad \text{braided Jacobi identity}$$

$$\{\varphi,\psi\,\chi\}_\star \ = \ \{\varphi,\psi\}_\star\,\chi \pm (R_\alpha\psi)\,\{R^\alpha\varphi,\chi\}_\star \qquad \text{braided Leibniz rule}$$

Braided BV Quantization

- Braided BV Laplacian $\Delta_{\rm BV} : {\rm Sym}_{\mathcal{R}} V[2] \longrightarrow ({\rm Sym}_{\mathcal{R}} V[2])[1]$:

$$\Delta_{\rm BV}(\mathbb{1}) = 0 = \Delta_{\rm BV}(\varphi) \quad , \quad \Delta_{\rm BV}(\varphi\,\psi) = \{\varphi,\psi\}_\star$$

$$\Delta_{\rm BV}(a\,b) = \Delta_{\rm BV}(a)\,b + (-1)^{|a|}\,a\,\Delta_{\rm BV}(b) + \{a,b\}_\star$$

$$\Delta_{\rm BV}\big(\varphi_1\cdots\varphi_n\big) = \sum_{i<j} \pm\,\langle\varphi_i, R_{\alpha_{i+1}}\cdots R_{\alpha_{j-1}}\varphi_j\rangle_\star \\ \times\ \varphi_1\cdots\varphi_{i-1}\,(R^{\alpha_{i+1}}\varphi_{i+1})\cdots(R^{\alpha_{j-1}}\varphi_{j-1})\,\varphi_{j+1}\cdots\varphi_n$$

  Implements *braided* Gaussian integration/Wick's Theorem (Oeckl '99)

- Satisfies $\ell_1^\star\,\Delta_{\rm BV} + \Delta_{\rm BV}\,\ell_1^\star = 0\ ,\ \Delta_{\rm BV}^2 = 0\ ,\ \Delta_{\rm BV}(S_{\rm int}^\star) = 0$

- $Q_{\rm BV}^2 = 0$ where $Q_{\rm BV} = \ell_1^\star + \{S_{\rm int}^\star, -\} + {\rm i}\,\hbar\,\Delta_{\rm BV}$

- Quantum observables $\big({\rm Sym}_{\mathcal{R}} V[2], Q_{\rm BV}\big)$ form a braided E_0-algebra

Braided Homological Perturbation Theory

- Propagators give braided strong deformation retracts of $V[1]^* \simeq V[2]$:

$$\big(H^\bullet(V[2]),0\big) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \big(V[2],\ell_1^\star\big) \circlearrowleft \gamma \qquad \pi\,\iota = \mathbb{1}\ ,\ \iota\,\pi - \mathbb{1} = \ell_1^\star\,\gamma + \gamma\,\ell_1^\star \quad \gamma^2 = 0\ ,\ \gamma\,\iota = 0\ ,\ \pi\,\gamma = 0$$

where $\pi,\iota = U_{\mathcal{F}}\Gamma(TM)$-equivariant , $\gamma = U_{\mathcal{F}}\Gamma(TM)$-invariant

- Observables: $$\big(\mathrm{Sym}_{\mathcal{R}}H^\bullet(V[1]),0\big) \underset{\Pi}{\overset{\mathcal{I}}{\rightleftarrows}} \big(\mathrm{Sym}_{\mathcal{R}}V[2],\ell_1^\star\big) \circlearrowleft \Gamma$$

- Homological Perturbation Lemma: With $U_{\mathcal{F}}\Gamma(TM)$-invariant $\delta = \{S^\star_{\rm int},-\}_\star + \mathrm{i}\,\hbar\,\Delta_{\rm BV}$, there is a braided strong deformation retract

$$\big(\mathrm{Sym}_{\mathcal{R}}H^\bullet(V[2]),\widetilde{\delta}\,\big) \underset{\widetilde{\Pi}}{\overset{\widetilde{\mathcal{I}}}{\rightleftarrows}} \big(\mathrm{Sym}_{\mathcal{R}}V[2],Q_{\rm BV}\big) \circlearrowleft \widetilde{\Gamma}$$

where $$\widetilde{\Pi} = \Pi\,(\mathbb{1}-\delta\,\Gamma)^{-1}\,\delta\,\Gamma = \Pi\circ\sum_{k=1}^\infty(\delta\,\Gamma)^k$$

- $\langle\varphi_1\cdots\varphi_n\rangle = \widetilde{\Pi}(\varphi_1\cdots\varphi_n) \in \mathrm{Sym}_{\mathcal{R}}H^\bullet(V[2])$ are (smeared) n-point correlation functions on vacua $H^\bullet(V[1])$ of the braided field theory

Noncommutative Scalar Field Theory

- $V = V_1 \oplus V_2$, $V_1 = V_2 = C^\infty(\mathbb{R}^4)$, Moyal-Weyl twist:

$$\ell_1^\star = \ell_1 = \Box + m^2 \qquad , \qquad \ell_3^\star(\phi_1, \phi_2, \phi_3) = \lambda\, \phi_1 \star \phi_2 \star \phi_3$$

- Braided field equations:

$$F_\phi^\star = \ell_1^\star(\phi) + \tfrac{1}{3!}\, \ell_3^\star(\phi, \phi, \phi) = (\Box + m^2)\, \phi + \tfrac{\lambda}{3!}\, \phi \star \phi \star \phi$$

- With cyclic inner product $\langle \phi, \phi^+ \rangle_\star = \int \mathrm{d}^4x\ \phi \star \phi^+$, action is:

$$S^\star = \int \mathrm{d}^4x\ \Big(\frac{1}{2}\, \phi \star (\Box + m^2)\, \phi + \frac{\lambda}{4!}\, \phi \star \phi \star \phi \star \phi \Big)$$

- **Standard** noncommutative scalar field theory is organised by a **braided** L_∞-algebra!

- Plane waves $e_k(x) = \mathrm{e}^{\,\mathrm{i}\, k \cdot x}$, $\langle e_k^*, e_p \rangle_\star = (2\pi)^4\, \delta^4(k - q)$

Noncommutative Scalar Field Theory

- Interactions: $S^{\star}_{\rm int} \;=\; \int_{k_1,\dots,k_4} V_{k_1,\dots,k_4}\, e^*_{k_1}\cdots e^*_{k_4} \;\in\; \mathrm{Sym}_{\mathcal{R}}\, V[2]$:

$$V_{k_1,\dots,k_4} \;=\; \mathrm{e}^{\,\mathrm{i}\sum_{I<J} k_I\times k_J}\,(2\pi)^4\,\delta^4(k_1+\cdots+k_4)$$

- Deformation retract: $H^{\bullet}(V[2]) \;=\; 0$ for $m^2 > 0$:

$$(0,0) \underset{0}{\overset{0}{\rightleftarrows}} \big(V[2],\ell_1\big) \circlearrowleft_{-G} \qquad G \;=\; \ell_1^{-1} \;=\; (\Box + m^2)^{-1}$$

- Correlation functions: $(\mathbb{C},0) \underset{\widetilde{\Pi}}{\overset{\widetilde{\mathcal{I}}}{\rightleftarrows}} \big(\mathrm{Sym}_{\mathcal{R}}\, V[2], Q_{\rm BV}\big) \circlearrowleft_{\widetilde{\Gamma}}$

$$\langle \phi(x_1)\star\cdots\star\phi(x_n)\rangle \;:=\; \sum_{k=1}^{\infty} \Pi\,\big(\mathrm{i}\,\hbar\,\Delta_{\rm BV}\,\Gamma + \{S^{\star}_{\rm int},-\}_{\star}\,\Gamma\big)^k\,(\delta_{x_1}\cdots\delta_{x_n})$$

where $\delta_{x_i}(x) \;=\; \delta^4(x-x_i)$; only $\Pi(\mathbb{1}) \;=\; 1$ is non-zero (as $\pi = 0$)

Noncommutative Scalar Field Theory

- **Example 1:** 4-point function of free braided scalar field ($\lambda = 0$):

$$\langle \phi(x_1) \star \cdots \star \phi(x_4) \rangle \; = \; (\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\Gamma)^2 (\delta_{x_1} \cdots \delta_{x_4})$$

$$= \; \langle \phi_1\,\phi_2 \rangle\,\langle \phi_3\,\phi_4 \rangle + \langle \phi_1\,R_\alpha \phi_3 \rangle\,\langle R^\alpha \phi_2\,\phi_4 \rangle + \langle \phi_1\,\phi_4 \rangle\,\langle \phi_2\,\phi_3 \rangle$$

where $\langle \phi_i\,\phi_j \rangle \; := \; -\,\mathrm{i}\,\hbar \displaystyle\int_k \frac{\mathrm{e}^{\,-\,\mathrm{i}\,k\cdot(x_i - x_j)}}{k^2 + m^2}$; Braided Wick's Theorem

- **Example 2:** 2-point function at 1-loop (order λ):

$$\langle \phi(x_1) \star \phi(x_2) \rangle \; = \; (\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\Gamma)^2\,\{S_{\mathrm{int}}^\star, \Gamma\,(\delta_{x_1}\,\delta_{x_2})\}_\star$$

$$= \frac{\hbar^2\,\lambda}{2} \int_{k_1, k_2} \frac{\mathrm{e}^{\,-\,\mathrm{i}\,k_1\cdot(x_1 - x_2)}}{(k_1^2 + m^2)^2\,(k_2^2 + m^2)} \; = \; -\,\mathrm{i}\,\hbar \int_k \frac{\mathrm{e}^{\,-\,\mathrm{i}\,k\cdot(x_1 - x_2)}}{k^2 + m^2 + \mathit{\Pi}_\star(k^2)}$$

identifies self-energy

$$\frac{\mathrm{i}}{\hbar}\,\mathit{\Pi}_\star \; = \; -\frac{\lambda}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{1}{p^2 + m^2}$$

No UV/IR mixing (at this order in this correlation function)